Influence of Pressure Gradients on Nonhomogeneous Dissipative Free Flows

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The influence of pressure gradients on the laminar mixing of two semi-infinite streams of different gases is studied. Constant density, constant transport properties, and arbitrary Schmidt number *Sc* are assumed. Solutions are obtained as a series in terms of a parameter *m* proportional to the momentum flux difference of the two streams. Concentration profiles are presented, in explicit forms, up to terms of order *m*. This approximation is satisfactory for an initial velocity ratio of about 0.5 when the *Sc* is of order one and is better or worse for low or high values of *Sc*. It is found that, within the limits of the present approximation, favorable pressure gradients improve the diffusion process; adverse gradients act in the opposite manner. Pressure-gradient effects become greater as the *Sc* and the initial velocity ratio decrease.

Nomenclature†

a^2	= Schmidt number Sc
b_r	= coefficients in velocity distribution [Eq. (2.9)]
c, C	= concentration
$g(x)$	= function defined by Eq. (1.5)
$i^0(t)$	= complementary error function
$i^{-1}(t)$	= $[2/(\pi)^{1/2}]e^{-t^2}$
m	= $(1 - U_{20}^2)/2$
p	= nondimensional pressure
u, v	= nondimensional velocity components in inner region
x, y	= Prandtl variables
$C_{ir}(\sigma)$	= functions defined by Eq. (2.4)
$G_{ir}(\sigma)$	= polynomials defined by Eq. (3.5)
$L(a, \sigma)$; $L^0(a, \sigma)$	= functions defined by Eqs. (3.2)
M	= mass entrained by diffusion
Re	= $L^+U_1^+(0)/\nu^+$
U, V	= nondimensional velocity components in outer region
$U_i(X)$	= velocities of two streams along X axis ($i = 1, 2$)
U_{21}	= U_2/U_1
U_{20}	= $U_2(0)/U_1(0)$
X, Y	= nondimensional space variables
$\alpha_{r,i}$; $\beta_{r,i}$; $\gamma_{r,i}$	= coefficients defined in Eq. (3.5)
$\zeta = U_1^2 - u^2$	= dependent variable defined by Eq. (2.1)
$\zeta_i(\sigma)$	= functions defined by Eq. (2.4)
ξ, η	= Von Mises variables, Eq. (2.1)
σ	= $\eta/2(\xi)^{1/2}$
$\psi(x, y)$	= nondimensional stream function

Introduction

THIS paper presents a contribution to the study of pressure-gradient effects on the characteristics of mixing. Some nonisobaric dissipative flow fields in the absence of solid boundaries already have been considered by a number of authors.[1–8] Apparently, however, this subject has received less attention than that of isobaric flow fields. A great deal more needs to be known about a variety of situations of actual, or possibly future, technological importance. It appears that no study has been made of the nonisobaric mixing of two semi-infinite streams of different gases, a subject to which this paper is specifically devoted. Only the simplest case, corresponding to the crudest possible approximations, will be considered.

The flow is assumed to be laminar and the initial conditions to be uniform. The variations of density, viscosity, and diffusion coefficients are neglected in the balance equations. This constant-property approximation is not new in the study of dissipative flow fields of binary mixtures[9] or of homogeneous gases.[10] In the subject case it implies 1) sufficiently small Mach number,[10] 2) not-too-large temperature differences between the two streams, and 3) sufficiently small difference between the molecular diameters of the two gases. The last additional requirement is needed to neglect the concentration dependence of the viscosity coefficient[11] (the first-order diffusion coefficient is independent of concentration), and it is met adequately by the following gases[11]: CO, N_2, NO, O_2, and A.‡ For other gases one presumably may still make the constant-viscosity approximation by defining a suitable average viscosity coefficient.

The pressure field will be assumed to be characterized by a streamwise analytic distribution, and the constant Sc will not be taken as equal to one. Only the diffusion equation will be explicitly and extensively considered; the velocity field in the same hypothesis has been solved already.[3,7] Results obtained could be extended to the turbulent case provided that such matters as suitable expressions for eddy transport coefficients are accepted. However, in view of the crude assumptions for the laminar case and of the many debatable opinions on the validity of these "turbulent models," the extension will not be pursued here. The solutions will be sought as a series in terms of the initial momentum flux difference of the two streams, and only the first two terms of the series will be given.

I. Basic Equations

Consider the nonisobaric flow field caused by two semi-infinite streams of different gases which interact starting from $X = 0$. The first-order potential flow (corresponding with the solution of the first Euler limit[12] or outer limit[13] of the

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† All of the dimensional quantities are indicated by a superscript plus.

‡ If the two streams have the same density, their temperatures must be in the inverse ratio of their molecular weights, and the equality of molecular diameters is again a sufficient condition for the equality of the first-order viscosity coefficients.

full set of balance equations) is characterized by a continuous pressure field and by piecewise continuous velocity and concentration fields, the plane $Y = 0$ being the discontinuity surface.

The first-order dissipative flow (corresponding with the solution of the first Prandtl limit[12] or inner limit[13] of the full set of balance equations) can be evaluated up to an indeterminacy of its orientation in the physical (X, Y) plane when the values of the first-order outer solution along the Y axis and along both sides of the discontinuity line are prescribed.

Let these data be

$$\left.\begin{array}{c} p_0(0, Y) = 1 \qquad p_0(X, 0) = p(X) \qquad X(0) = 1 \\ V(0, Y) = 0 \qquad V(X, 0) = 0 \\ U(X, 0^+) = U_1(X) \qquad U(X, 0^-) = U_2(X) \\ U(0, Y) = U_1(0) = 1 (Y > 0) \\ U(0, Y) = U_2(0) = U_{20} < 1 (Y < 0) \\ C(0,Y) = C(X, 0^+) = 0 (Y > 0) \\ C(0, Y) = C(X, 0^-) = 1 (Y < 0) \end{array}\right\} \quad (1.1)$$

where $p(X)$ and $U_i(X)$ are analytic functions satisfying the requirements imposed by the first-order outer balance equations

$$\begin{array}{cc} p + \frac{1}{2}\rho_i U_i^2 = \text{const} \qquad i = 1, 2 \\ U_1^2(X) - U_2^2(X) = 1 - U_{20}^2 = \text{const} \end{array} \quad (1.2)$$

The isobaric case corresponds with $U_1(X) = 1$; $U_2(X) = U_{20}$.

The jump of the momentum flux across the discontinuity line $Y = 0$ is constant. However, because of the smaller "inertia" of the slower stream (subscript 2), the velocity ratio varies according to the relation

$$U_{21}^2(X) = U_2^2(X)/U_1^2(X) = 1 - (1 - U_{20}^2)/U_1^2(X) \quad (1.3)$$

For favorable pressure gradients ($p_x < 0$), the surface $Y = 0$ tends toward a vortical surface of vanishing strength. For adverse pressure gradients ($p_x > 0$), the slower stream decelerates faster (U_{21} decreases), and conditions are reached for which the present analysis is invalidated. In terms of nondimensional quantities, the equations for the first-order dissipative flow field are, in the constant-property approximation,

$$\left.\begin{array}{c} u_x + v_y = 0 \\ uu_x + vu_y = u_{yy} + U_1 U_{1x} \\ uc_x + vc_y = (1/Sc)\, c_{yy} \end{array}\right\} \quad (1.4)$$

The quantities appearing herein are related to the physical velocity components u^+ and v^+ (pertinent to the inner expansion) and space variables X^+ and Y^+ by

$$\left.\begin{array}{cc} u = \dfrac{u^+}{U_1^+(0)} & v = \dfrac{v^+}{U_1^+(0)} Re^{1/2} + \dfrac{u^+}{U_1^+(0)} g'(x) \\ x = \dfrac{X^+}{L^+} & y = \dfrac{Y^+}{L^+} Re^{1/2} + g(x) \end{array}\right\} \quad (1.5)$$

where $g(x)$ is an arbitrary function with $g(0) = 0$.

The imposition of the physically appropriate boundary condition for the velocity component $v^+(x, y)$ also requires, in general, a known second-order solution of the outer flow.[14] However, the invariance of the first-order inner equations, under the transformations defined by Eq. (1.5), introduces a mathematical degree of freedom[3] [the function $g(x)$]. It then will be possible to impose any convenient boundary condition on the function $v(x, y)$ and to satisfy the appropriate physical boundary condition on $v^+(x, y)$, a posteriori, by means of the function $g(x)$. If $v(x, 0) = 0$, the equation of the streamline through the origin is $y = 0$ in the (x, y) plane, and

$$Y^+/L^+ = -Re^{-1/2}g(x) \qquad g(0) = 0$$

in the physical (X^+, Y^+) plane. The shape of this streamline can be determined only when the second-order solution of the outer flow is known, and this in turn requires a specification of the entire first-order outer flow field. In the present paper $g(x)$ shall be left undetermined; therefore, the first-order inner flow field is determined only up to its orientation in the physical plane. This orientation is the only feature differentiating dissipative regions caused by first-order outer flow fields with the same flow properties along the coordinate axes. The boundary and initial conditions for Eq. (1.4) then are, according to Eq. (1.1)

$$\left.\begin{array}{cc} u(0, y) = \begin{cases} 1 \ (y > 0) \\ U_{20}(y < 0) \end{cases} & c(0, y) = \begin{cases} 0 \ (y > 0) \\ 1 \ (y < 0) \end{cases} \\ y \to \infty. \ \ u(x, y) \to U_1(X) & c(x, y) \to 0 \\ y \to -\infty: \ \ u(x, y) \to U_2(X) & c(x, y) \to 1 \\ v(0, y) = 0 & v(x, 0) = 0 \end{array}\right\} \quad (1.6)$$

II. Reduction of Basic Equations

The Von Mises variables are

$$\left.\begin{array}{cc} \xi = \displaystyle\int_0^x U_1(t)dt & \eta = \psi(x, y) \\ \zeta(\xi, \eta) = U_1^2(x) - u^2(x, y) & c(x, y) = C(\xi, \eta) \end{array}\right\} \quad (2.1)$$

where ψ is the nondimensional stream function defined by

$$\psi_y = u \qquad \psi_x = -v$$

The Jacobian of the transformation equals (uU_1), so that the transformation is proper as long as neither $U_1(x)$ nor $u(x, y)$ vanishes. In terms of the new variables, Eq. (1.4) reads as

$$\begin{array}{c} \zeta_\xi = [1 - (\zeta/U_1^2)]^{1/2}\zeta_{\eta\eta} \\ ScC_\xi = \{[1 - (\zeta/U_1^2)]^{1/2}C_\eta\}_\eta \end{array} \quad (2.2)$$

and the boundary and initial conditions become

$$\left.\begin{array}{cc} \zeta(0, \eta) = \begin{cases} 0(\eta > 0) \\ 1 - U_{20}^2(\eta < 0) \end{cases} & \begin{array}{c} \zeta(\xi, +\infty) = 0 \\ \zeta(\xi, -\infty) = 1 - U_{20}^2 \end{array} \\ C(0, \eta) = \begin{cases} 0 \ (\eta > 0) \\ 1 \ (\eta < 0) \end{cases} & \begin{array}{c} C(0, +\infty) = 0 \\ C(0, -\infty) = 1 \end{array} \end{array}\right\} \quad (2.2a)$$

Once the functions ζ and C have been determined in the (ξ, η) plane, the pertinent flow properties in the (x, y) plane can be obtained from

$$\left.\begin{array}{c} x(\xi, \eta) = \displaystyle\int_0^\xi U_1^{-1}(t)dt \qquad U_1(0) = 1 \\ y(\xi, \eta) = U_1^{-1}(\xi) \displaystyle\int_0^\eta \left\{1 - \left[\frac{\zeta(\xi, t)}{U_1^2(\xi)}\right]\right\}^{-1/2} dt \\ u(x, y) = [U_1^2(\xi) - \zeta(\xi, \eta)]^{1/2} \\ v(x, y) = u(x, y)U_1(x) \dfrac{\partial y(\xi, \eta)}{\partial \xi} = \\ -yu \dfrac{dU_1}{dx} + \dfrac{u}{2} \displaystyle\int_0^\eta \frac{[\zeta(\xi, t)/U_1^2(\xi)]_\xi \, dt}{\{1 - [\zeta(\xi, t)/U_1^2(\xi)]\}^{3/2}} \\ c(x, y) = C(\xi, \eta) \end{array}\right\} \quad (2.3)$$

Solutions of Eq. (2.2) for arbitrary Sc will be sought in the form

$$\begin{array}{c} \zeta(\xi, \eta) = \displaystyle\sum_{i=1} m^i \zeta_i(\xi, \eta) \\ C(\xi, \eta) = \displaystyle\sum_{i=0} m^i C_i(\xi, \eta) \end{array} \quad (2.4)$$

where the parameter m is defined as

$$m = (1 - U_{20}^2)/2 \tag{2.5}$$

and is a measure of the constant-momentum flux jump. The first summation starts from $i = 1$ since, as is apparent from Eq. (2.2), the zeroth-order solution for ζ is $\zeta_0 = 0$. Because of the constant-property assumption, the momentum equation is not coupled with the diffusion equation. General explicit expressions for the ζ_i's up to $i = 2$ and for some classes of analytic pressure gradients are given in Refs. 3 and 7. Only the first two terms in the diffusion equation will be considered. The required first-order solution of the momentum equation is given by[3]

$$\zeta_1 = i^0(\sigma) \qquad \sigma = \eta/2\xi^{1/2} \tag{2.6}$$

where $i^0(\sigma)$ is the complementary error function.

The equations for the first two terms of the second series are

$$ScC_{0\xi} = C_{0\eta\eta} \tag{2.7}$$

$$Sc\, C_{1\xi} = C_{1\eta\eta} - [1/2U_1^2(\xi)][\zeta_1 C_{0\eta}]_\eta \tag{2.8}$$

subject to the conditions

$$\left.\begin{array}{l} C_0(0, \eta) = \begin{cases} 0 & (\eta > 0) \\ 1 & (\eta < 0) \end{cases} \\ C_0(\xi, +\infty) = 0 \qquad C_0(\xi, -\infty) = 1 \end{array}\right\} \tag{2.7a}$$

$$C_1(0, \eta) = C_1(\xi, \pm\infty) = 0 \tag{2.8a}$$

The zeroth-order approximation $C_0(\xi, \eta)$ corresponds with conditions of isovel mixing. A solution to this problem in the isobaric case without the assumption of constant properties was given in Ref. 16.

The first-order concentration field depends explicitly upon the pressure field. The following classes of outer velocity distributions are considered:

$$\frac{1}{U_1^2(\xi)} = \sum_{r=0}^{n} b_r \xi^r \qquad b_0 = 1 \tag{2.9}$$

where n is an arbitrary integer, and the otherwise arbitrary constant b_r's are such that the summation is never zero in the considered range of ξ. The corresponding expression for the pressure gradient is

$$p_x(x, 0) = -2 \sum_{r=1}^{n} r b_r \xi^{(r-1)} \bigg/ \left(\sum_{r=1}^{n} b_r \xi^r \right)^5$$

which is an analytic function in view of the limitation imposed on Eq. (2.9). The isobaric case corresponds with $n = 0$ $(U_1 = 1)$.

If $Sc = a^2$, the solution of Eq. (2.7) is

$$C_0 = \tfrac{1}{2} i^0(a\sigma) \qquad \sigma = \eta/2\xi^{1/2} \tag{2.10}$$

Since Eq. (2.8) is linear and ζ_1 does not depend on $U_1^2(\xi)$, C_1 can be expressed as

$$C_1(\xi, \eta) = \sum_{r=0}^{n} \frac{ab_r}{4} \xi^r C_{1r}(\sigma) \tag{2.11}$$

where, accounting for Eqs. (2.6, 2.8, and 2.10), the $C_{1r}(\sigma)$ are found to satisfy the equation

$$\left.\begin{array}{r} C_{1r}'' + 2a^2\sigma C_{1r}' - 4a^2 r C_{1r} = i^{-1}(\sigma)i^{-1}(a\sigma) + \\ 2\sigma a^2 i^0(\sigma) i^{-1}(a\sigma) \\ C_1r(\pm\infty) = 0 \end{array}\right\} \tag{2.12}$$

where primes denote differentiation with respect to σ, and $-i^{-1}(t)$ is the derivative of the function $i^0(t)$.

The formal solution of Eq. (2.12) is not difficult to obtain, e.g., the method of undetermined constants could be used. Also, the general solution of Eq. (2.8) could be given formally in terms of the appropriate Green's function. In both cases, however, a number of integrals, which are rather difficult to evaluate in closed form, would be obtained. In the next section, the solution of Eq. (2.12) is expressed explicitly in terms of known functions related to the complementary error function by means of a method originally developed in connection with the solution of the second approximation to the nonisobaric velocity field.[7]

III. Solution of the First-Order Diffusion Equation

The method that will be applied to obtain the "normal solutions" of second-order ordinary differential equations with essential singularities at infinitive can be considered a generalization of the Thomé method.[15] Briefly, the method hinges upon the following considerations. Assume it is possible to identify, a priori, the nature of the essential singularities of the function C_{1r} at ∞. If the set of functions $Z_i(\sigma)$, $i = 1, \ldots, k$ satisfies the boundary conditions for $C_{1r}(\sigma)$ and contains all of the essential singularities of $C_{1r}(\sigma)$ (and only these), then the solution of Eq. (2.12) can be expressed as

$$C_{1r}(\sigma) = \sum_{i=1}^{k} G_{ir}(\sigma) Z_i(\sigma) \tag{3.1}$$

where k depends upon the number of singularities of $C_{1r}(\sigma)$, and the $G_{ir}(\sigma)$ are polynomials in σ (i.e., functions with, at most, poles at $|\sigma| = \infty$).

A necessary condition for the representation (3.1) to be valid is that, upon substitution in Eq. (2.12), one obtains a system of equations for the $G_{ir}(\sigma)$ in which all $Z_i(\sigma)$ drop out. The sufficient condition is that each resulting equation admits at least one solution free of essential singularities at infinity [if such is not the case, the set $Z_i(\sigma)$ does not contain all of the singularities of $C_{1r}(\sigma)$].

The crucial point in the application of this method lies, of course, in the identification of the set of functions $Z_i(\sigma)$. With some practice, however, this is not a difficult task, and the procedure presents the advantage of obtaining explicit, closed-form solutions without going through difficult evaluations of integrals. In the present case it can be shown that the essential singularities of $C_{1r}(\sigma)$ are all contained in the set

$$\begin{array}{ccc} i^{-1}(a\sigma) & i^{-1}[\sigma(1 + a^2)^{1/2}] & \\ i^0(a\sigma) & L^0(a, \sigma) & L(a, \sigma) \end{array}$$

where

$$\left.\begin{array}{l} L^0(a, \sigma) = i^0(\sigma)i^{-1}(a\sigma) = -(d/d\sigma)[L(a, \sigma)] \\ L(a, \sigma) = \int_\sigma^\infty i^0(t)i^{-1}(at)dt \end{array}\right\} \tag{3.2}$$

The properties of the complementary error function $i^0(t)$ and of its first derivative $i^{-1}(t)$ are well known[17]; their values are tabulated in Ref. 18. The function $L(a, \sigma)$ belongs to the classes of functions introduced, studied, and tabulated in Ref. 19. For $a = 0$, it reduces to the first integral of the complementary error function[17]; for $a = 1$ $(Sc = 1)$ and $a = (2)^{1/2}(Sc = 2)$, it reduces to functions studied in Refs. 20 and 21, respectively. In the present work the following identities will be needed[19]:

$$\left.\begin{array}{l} L(a, \infty) = 2/a \qquad L^0(a, 0) = i^{-1}(0) = 2/(\pi)^{1/2} \\ L(a, 0) = \dfrac{2}{a\pi}\, tn^{-1}(a) = \dfrac{1}{a}\left[1 - \dfrac{2}{\pi}\, tn^{-1}\left(\dfrac{1}{a}\right)\right] \end{array}\right\} \tag{3.3}$$

Values of the function $(1 + a^2)^{-1/2}L(a, \sigma)$ are reported in Table 1 for the following indicative values of a: $(2)^{1/2}$, $(3)^{1/2}$, 2, and 10. From the identity[19]

$$L(a, \sigma) = \frac{1}{a}\, i^0(a\sigma)i^0(\sigma) - \frac{1}{a^2}\, L\left(\frac{1}{a}, a\sigma\right)$$

and from the tables of Ref. 18, values of $L(a, \sigma)$ thus can be obtained for the following indicative values of Sc: 0.01, 0.25, $\frac{1}{3}$, 0.5, 1, 2, 3, 4, and 100. Values for $\sigma < 0$ are obtained from the relation[19]

$$L(a, -\sigma) = (2/a)[1 - i^0(a\sigma)] + L(a, \sigma)$$

The boundary conditions [Eq. (2.12)] lead to the following representation for $C_{1r}(\sigma)$:

$$C_{1r}(\sigma) = -\frac{\sigma i^{-1}(a\sigma)}{2(r+1)} + G_{1r}(\sigma)[L^0(a, \sigma) - i^{-1}(a\sigma)] + G_{2r}(\sigma)\frac{2}{(\pi)^{1/2}} i^{-1}[\sigma(1+a^2)^{1/2}] + G_{3r}(\sigma)\left[L(a, \sigma) - \frac{i^0(a\sigma)}{a}\right] \quad (3.4)$$

where

$$\left.\begin{aligned} G_{1r}(\sigma) &= (1+a^2)^{(r+1)}D_r \sum_{t=0}^{r-1} a^{2t}\alpha_{r,\,2t+1}\sigma^{2(t+1)} \\ G_{2r}(\sigma) &= \frac{1}{4(r+1)} + (1+a^2)D_r \sum_{t=0}^{r-1} (1+a^2)^t \gamma_{r,\,2t}\sigma^{2t} \\ G_{3r}(\sigma) &= (1+a^2)^{(r+1)}D_r \sum_{t=0}^{r} a^{2t}\beta_{r,2t}\sigma^{2t} \end{aligned}\right\} \quad (3.5)$$

Indeed, for all of the functions $Z_i(\sigma)$, which multiply the $G_{ir}(\sigma)$, it is, for any $s > 0$, $\sigma^s Z_i \to 0$ when $\sigma \to \pm\infty$ as a consequence of the asymptotic behavior of $i^0(t)$, $i^{-1}(t)$, $L^0(t)$, and $L(t)$; and because, when $\sigma \to -\infty$, $i^0 \to 2$, and $L(a, \sigma) \to (2/a)$ [Eq. (3.3)].

The coefficients appearing in Eq. (3.5) satisfy the system of algebraic equations obtained by substituting Eq. (3.4) into Eq. (2.12), by equating to zero the coefficients of the functions that multiply the $G_{ir}(\sigma)$, and by looking for polynomial solutions of the resulting system of ordinary differential equations. Details of the procedure are omitted, and only the final results are given. The coefficients $\beta_{r,i}$; $\alpha_{r,i}$, and D_r are given explicitly by

$$\beta_{r,\,2t} = \frac{2^{2t}r!}{(2t)!(r-t)!} \qquad 0 \le t \le r$$

$$-\alpha_{r,\,2t+1} = \frac{2^{(2t+1)}r!(r+t)!}{(2t+1)!(2t)!}\sum_{s=0}^{r-t-1}\frac{1}{s!(2r-s)!} \qquad (0 \le t \le r-1)$$

$$D_r = \frac{(1+2r)/2(r+1)}{(1+a^2)^r(1-2\alpha_{r,1}) + 2(1+a^2)(\gamma_{r,2} - \gamma_{r,0}) - 4ra^2\gamma_{r,0}}$$

whereas the coefficients $\gamma_{r,i}$ must be determined from the recurrence formula

$$2(t+1)(2t+1)(1+a^2)\gamma_{r,\,2(t+1)} - 2[(4t+1) + (2t+2r+1)a^2]\gamma_{r,\,2t} + 4\gamma_{r,\,2(t-1)} = a^{2t}(1+a^2)^{r-t}\{2(2t+1)\alpha_{r,\,2t+1} - [2(1+a^2)/a^2]\alpha_{r,\,2t-1} - \beta_{r,\,2t}\} \qquad (1 \le t \le r) \quad (3.6)$$

where all of the coefficients are zero when their index is negative (for instance, $\alpha_{r,\,2t-1} = 0$ for $t = 0$) and $\gamma_{r,\,2t} = 0$ for $t \ge r$.

The solution of Eq. (3.6) is straightforward since the terms are uncoupled. Coefficients for r = 0, 1, 2, and 3 are listed in Table 2. Explicit expressions for $C_{10}(\sigma)$ (isobaric case) and $C_{11}(\sigma)$ are

$$C_{10}(\sigma) = -\frac{\sigma}{2}L^0(a, \sigma) + \frac{i^{-1}[\sigma(1+a^2)^{1/2}]}{2(\pi)^{1/2}} + \frac{1+a^2}{2}\left[L(a, \sigma) - \frac{i^0(a, \sigma)}{a}\right]$$

$$C_{11}(\sigma) = -\sigma\frac{2+3(1+a^2)^2}{8}L^0(a, \sigma) + \frac{3}{8}(1+a^2)^2\sigma i^{-1}(a\sigma) + \frac{3a^2+5}{8(\pi)^{1/2}}i^{-1}[\sigma(1+a^2)^{1/2}] + \frac{3}{8}(1+a^2)^2(1+2a^2\sigma^2)\left[L(a, \sigma) - \frac{i^0(a, \sigma)}{a}\right] \quad (3.7)$$

In particular, for $Sc = 1 (a = 1)$, the following is obtained:

$$\left.\begin{aligned} C_{10}(\sigma) &= [i(\sigma)i^{-1}(\sigma)/2] + i^0(\sigma)\{[i^0(\sigma)/2] - 1\} \\ C_{11}(\sigma) &= (i^{-1}(\sigma)/2)\{i^{-1}(\sigma) + \sigma[\tfrac{3}{2} - \tfrac{7}{4}i^0(\sigma)]\} + \tfrac{3}{2}(1+2\sigma^2)i^0(\sigma)\{[i^0(\sigma)/2] - 1\} \end{aligned}\right\} \quad (3.8)$$

where $i(t) = [\frac{1}{2}i^{-1}(t) - ti^0(t)]$ is the first repeated integral of the complementary error function.[17]

Table 1 **Table of function** $(1 + Sc)^{-1/2}\int_{\sigma}^{\infty} i^0(t)i^{-1}[(Sc)t^{1/2}]dt$

$(1 + Sc)^{1/2}\sigma$	$Sc = 2$	$Sc = 3$	$Sc = 4$	$Sc = 100$
0.0	0.24829	0.19245	0.15760	0.00932
0.1	0.21198	0.16510	0.13566	0.00821
0.2	0.17855	0.13972	0.11519	0.00714
0.3	0.14832	0.11660	0.09642	0.00612
0.4	0.12145	0.09582	0.07955	0.00517
0.5	0.09801	0.07771	0.06466	0.00430
0.6	0.07792	0.06202	0.05176	0.00352
0.7	0.06100	0.04875	0.04079	0.00284
0.8	0.04702	0.03771	0.03165	0.00225
0.9	0.03568	0.02872	0.02416	0.00176
1.0	0.02664	0.02151	0.01814	0.00135
1.1	0.01957	0.01585	0.01340	0.00102
1.2	0.01414	0.01149	0.00973	0.00076
1.3	0.01005	0.00819	0.00695	0.00055
1.4	0.00702	0.00574	0.00488	0.00039
1.5	0.00482	0.00395	0.00337	0.00028
1.6	0.00326	0.00268	0.00228	0.00019
1.7	0.00216	0.00178	0.00152	0.00013
1.8	0.00144	0.00116	0.00100	0.00009
1.9	0.00091	0.00075	0.00064	0.00006
2.0	0.00057	0.00047	0.00040	0.00004
2.1	0.00035	0.00029	0.00025	0.00002
2.2	0.00022	0.00018	0.00015	0.00001
2.3	0.00013	0.00011	0.00009	0.00000
2.4	0.00008	0.00007	0.00005	0.00000
2.5	0.00005	0.00004	0.00003	0.00000

IV. Analysis of Results

The solutions obtained can be used to discuss the combined influence of pressure gradients and arbitrary but constant Sc.

Characteristics of the Dissipative Region

The concentration profiles in the Von Mises plane are given by

$$C(\xi, \eta) = \frac{i^0(a\sigma)}{2} + \frac{m(Sc)^{1/2}}{4}\sum_{r=0}^{n} b_r\xi^r C_{1r}(\sigma) \quad (4.1)$$

where the $C_{1r}(\sigma)$ are given by Eq. (3.4), and the coefficients b_r describe the velocity (or pressure) field [Eq. (2.9)]. For $n = 0$ ($b_0 = 1$), the field is isobaric. The equation of the streamline through the origin in the Von Mises plane is $\eta = 0$. The distribution of concentration along this streamline is, from Eqs. (3.4, 3.5, and 3.3),

$$C(\xi, 0) = \frac{1}{2} + \frac{m(Sc)^{1/2}}{4} \sum_{r=0}^{n} b_r \xi^r C_{1r}(0) = \frac{1}{2} + \frac{m(Sc)^{1/2}}{4\pi} \sum_{r=0}^{n} b_r \xi^r \left\{ \frac{1}{r+1} + 2\,(1 + Sc) D_r \times \left[2\gamma_{r,0} - \frac{(1+Sc)^r}{(Sc)^{1/2}}\, tn^{-1}(Sc)^{-1/2} \right] \right\} \quad (4.2)$$

The mass flux $j(\xi, 0) = j^+(\rho, 0)/\rho^+ U_1^+(0)$ along the dividing streamline and the mass $M = M^+/\rho^+ U_1^+(0) L^+$ of component one diffused (per unit time and per unit length normal to the plane of the motion) into the region above the dividing streamline are expressed by

$$Sc(Re)^{1/2} j(\xi, 0) = -c_y(x, 0) = -[u(\xi, 0)/2(\xi)^{1/2}]\{[\partial C(\xi, \sigma)/\partial\sigma]\}_{\sigma=0}$$

$$(Re)^{1/2} M = \int_0^\infty C(\xi, t) dt = (Re)^{1/2} \int_0^\xi j(t, 0) dt$$

From Eqs. (2.6) and (3.3–3.5), we obtain, to within terms of the order m,

$$\left.\begin{aligned} 2[\pi \xi Re Sc]^{1/2} j(\xi, 0) &= 1 - \frac{m}{4} \sum_{r=0}^{n} \frac{2r+1}{r+1} b_r \xi^r \\ [\pi \xi^{-1} Re Sc]^{1/2} M &= 1 - \frac{m}{4} \sum_{r=0}^{n} \frac{b_r \xi^r}{(r+1)} \end{aligned}\right\} \quad (4.3)$$

Zeroth-Order Approximation

This approximation corresponds to isovel mixing. Since the density is assumed to be constant, the solution to this approximation corresponds to the one-dimensional unsteady diffusion as viewed by an observer moving with the constant velocity common to all of the fluid particles. (The same conclusion cannot be reached if the density of the fluid is not constant since, in this case, the y component of the velocity is no longer identically zero, and the fluid particles move with different vectorial velocities.) This simple, unsteady diffusion problem is so well known that any additional remarks are unnecessary.

First-Order Approximation

An idea on the nature of the accuracy given by Eq. (4.1) can be obtained when the case $Sc = 1$, $p_x = 0$ is considered, for then C is known "exactly" to the same order accuracy with which u is known

$$C = (1 - u)/(1 - U_{21}) \quad (4.4)$$

(as long as $U_{21} \neq 1$). The solution for u, up to terms of order m^2, is[7]

$$u = [1 - m\zeta_1 - m^2\zeta_{20}]^{1/2} = [1 - mi^0(\sigma) - (m^2/4) i(\sigma) i^{-1}(\sigma)] \quad (4.5)$$

Substituting this expression and the analogous one for U_{21} into Eq. (4.4), developing the square roots in terms of m, and neglecting terms of order m^2 or higher, yield

$$2C = \zeta_1 + m[(\zeta_1^2 + 4\zeta_{20}/4) - (\zeta_1/2)]$$

which coincides with the result given by Eq. (4.1). Thus, at least for $Sc = 1$ (and it will be seen later that this qualification is essential), the accuracy of the solution given by Eq. (4.1) is comparable with that obtained from the second-order solution of the velocity field.

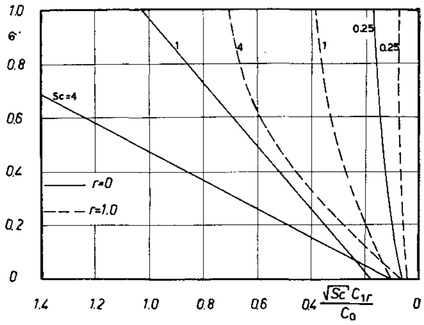

Fig. 1 Ratio between first- and zeroth-order contributions to the concentration profiles.

When $p_x \neq 0$, an integral of the diffusion equation cannot be obtained; even for $Sc = 1$, it is necessary to resort to other reasonings to estimate the range of validity of the present solutions.

The approximation given by Eq. (4.1) is not uniformly valid, both m-wise and coordinate-wise, in Sc. The first statement is verified by Fig. 1, where the ratios $[(Sc)^{1/2}C_{1r}/C_0]$ are plotted against σ for $r = 0$ and $r = 1$. Figure 1 shows that, in both cases, the ratio of the first- to zeroth-order approximation for $Sc = 4$ is roughly twice as large as that for $Sc = 1$ for finite positive values of σ. (The same trend was also found for higher values of r.) This implies that the values of m, for which results are accurate, decrease as the Sc increases (and vice versa). On the other hand, it can be shown that $(C_{10}/C_0) \to (1 + Sc)/[1 + (Sc)^{1/2}]$, $(C_{11}/C_0) \to 0$ as $\sigma \to \infty$ and that, because of the asymptotic behavior of $i^0(\sigma)$, $C_0(\sigma) \to 0$ as $\sigma \to \infty$ stronger than any power of Sc. Therefore, the series (4.1) is not uniformly valid in Sc coordinate-wise; the accuracy gets worse in the region of lower velocity. (This, however, is a common feature of all of the types of series solutions for mixing problems.)

More definite quantitative statements on the accuracy of the solution cannot be made for the lack of either exact solutions or of higher-order terms in the series (2.12). However, on the basis of the remarks made for the case $Sc = 1$, and in view of the findings of Ref. 3 concerning the accuracy of the velocity profiles, it may be argued that the present solution is satisfactory down to the values of the velocity ratio of about 0.5 when the Sc is of order one. Smaller values of Sc tend to improve the accuracy, whereas larger values act in the opposite manner. With these limitations in mind we can proceed to the discussion of the more important features of the solutions obtained.

For nonisobaric flow fields, the linearized velocity profiles are similar [see (Eq. 2.6)], whereas the "linearized" concentration profiles are not. Furthermore, contrary to what happens for the linearized velocity profiles, the concentration profiles exhibit an explicit dependence on the pressure gradient even in the Von Mises plane. All of this can be understood by looking at it from a thermodynamic point of view. To within terms of order m there is no entropy production caused by diffusion of momentum. One thus expects to find an appropriate representation of the dynamic

Table 2 Coefficients of polynomials $G_{ir}(6)$ [Eq. (3.5)]

r	D_r	α_1	α_3	α_5	β_0	β_2	β_4	β_6	γ_0	γ_2	γ_4
0	$\frac{1}{2}$	...	...	...	1	...	...	...	0	...	...
1	$\frac{3}{8}$	-1	...	...	1	2	...	...	$\frac{1}{2}$	...	...
2	$\frac{5}{16}$	$-\frac{5}{3}$	$-\frac{2}{3}$	...	1	4	$\frac{4}{3}$	...	$(5 + 3a^2)/6$	$a^2/3$	...
3	$\frac{105}{384}$	$-\frac{11}{5}$	$-\frac{28}{15}$	$-\frac{4}{15}$	1	6	4	$\frac{8}{15}$	$(33 + 40a^2 + 15a^4)/30$	$[2a^2(7 + 6a^2)]/15$	$2a^4/15$

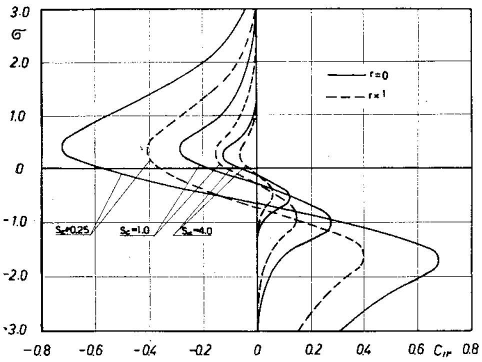

Fig. 2 First-order functions C_{1r}.

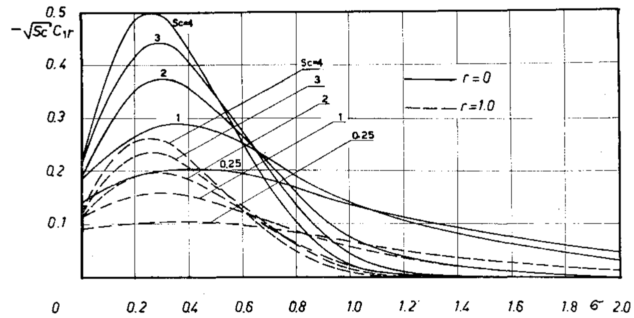

Fig. 3 First-order correction to the purely diffusive concentration profile.

state of the system that is independent of history. (It was indeed shown in Ref. 3 that the velocity profile at any station is uniquely identified for given initial momentum flux difference by the velocity ratio prevailing at that station.) This cannot be true for concentration profiles because the entropy production caused by diffusion of mass is not zero, even to within the first two terms of the series (2.12).

The functions C_{1r}'s are shown in Fig. 2 for $r = 0$, isobaric case, and $r = 1$ for indicative values of the Sc. The contributions corresponding with the several terms of the series that gives the distribution of $U_1(\xi)$ [Eq. (2.9)] are not symmetrical with respect to the dividing streamline, are all of the same sign, and decrease as r increases. This behavior is shown in the diagram only for $r = 0$ and $r = 1$, but it was also found for $r = 2$ and $r = 3$. Thus, the sign of the effects caused by the successive derivatives of the pressure field depends only upon the sign of b_r.

The Sc has a moderately strong influence on the corrections $[(Sc)^{1/2}C_{1r}]$ to the "purely diffusive" profile, as can be seen from Fig. 3, where this quantity is plotted against σ for values of the Sc ranging from 0.25 to 4. This influence is qualitatively the same for isobaric ($r = 0$) and nonisobaric fields.

The quantities $[(Sc)^{1/2}C_{1r}(0, \xi)]$ computed from Eq. (4.2) are plotted in Fig. 4 against Sc for $r = 0, 1, 2$. From this figure and from Eq. (4.2), it can be deduced that the concentration along the dividing streamline 1) is equal to $\frac{1}{2}$ for the purely diffusive case ($m = 0$); 2) is constant but less than $\frac{1}{2}$ for isobaric flows (this reduction for given m decreases as Sc increases); and 3) is not constant for nonisobaric fields. The contributions associated with each b_r are still negative, but as r increases they show a minimum for a value of Sc that decreases as r increases.

The particularly simple forms of Eq. (4.3) are worth noticing. The evaluation of the mass flux along the dividing streamline and of the mass of the component 1 diffused into the upper stream does not require the solution of the recurrence equation [Eq. (3.6)]. To within first-order terms in m, both quantities depend upon Sc only through the "scaling factor" $(Sc)^{1/2}$, which is independent of the particular pressure field considered. For given m, the contributions from the terms in b_r are all negative and are weighed by means of factors, which increase with r for the mass flux $j(\xi, 0)$ and decrease with r for the mass diffused M. The over-all effect of a given pressure field obviously depends upon the sign and magnitude of the coefficients b_r.

Some quantitative opinions can be obtained by considering the following indicative velocity field:

$$U_1^{-2}(\xi) = 1 + b_1\xi \qquad U_1(x) = [1 + (3/2)b_1x]^{-1/3} \tag{4.6}$$

corresponding with a pressure gradient of the type

$$p_x = (b_1/2)[1 + (3/2)b_1x]^{-5/3} \tag{4.7}$$

The sign of b_1 determines the nature of the pressure gradient (adverse for $b_1 > 0$ and favorable for $b_1 < 0$), and the magnitude of b_1 determines the intensity of the pressure gradient. The limiting situations $U_{21} = 1$ (discontinuity surface of vanishing strength) and $U_{21} = 0$ (slower stream brought to rest) correspond with the maximum effects of favorable and adverse pressure gradients, respectively. The first situation is attained for $b_1\xi = -1$, independently of the value of m; the second for $b_1\xi = (1 - 2m)/2m$.

The values of the concentrations $C(\xi, 0)$ along the dividing streamline are reported in Table 3 for $m = 0.2$ and $m = 0.4$ and for three indicative values of Sc. For each value of m the four columns show, in this order, values of $C(\xi, 0)$ in terms of $b_1\xi$; for $U_{21} = 1$ (i.e., $b_1\xi = -1$); $U_{21} = U_{20}$ (i.e., isobaric case); and $U_{21} = 0$ (i.e., $b_1\xi = (1 - 2m)/2m$). The values listed in the first three columns are attained for given b_1 at the same station in the physical plane. On the other hand, the condition $U_{21}^2 = 0$ is reached farther downstream for $m = 0.2$ than for $m = 0.4$. The following comments are in order:

At any given physical station x (i.e., for given $b_1\xi$ and given b_1) the pressure-gradient effects increase as the initial velocity ratio U_{20} and Sc decrease.

The values for $U_{21}^2 = 1$ clearly exhibit the "history dependence" since they differ from the purely diffusive value 0.5 that would prevail if U_{21} had always been equal to one from the origin of mixing. The "history dependence" is stronger for low values of the initial velocity ratio and Sc.

Along the streamline through the origin, and to within terms of order m, the influence of pressure gradients amounts to few percents, and so the isobaric values may often constitute a satisfactory approximation.

The expression for the mass diffused into the upper stream is, from Eq. (4.3),

$$(\pi Re\ Sc)^{1/2}M = (\xi)^{1/2}\{1 - (m/4)[1 + (b_1\xi/2)]\}$$

Table 3 Concentration $C(\xi,0)$ along the dividing streamline for the pressure field given by Eq. (4.7)

	$m = 0.2$; $U_{20} = 0.775$				$m = 0.4$; $U_{20} = 0.447$			
		Values for				Values for		
Sc	General expression	$U_{21} = 1$ ($p_x < 0$)	$U_{21} = U_{20}$ ($p_x = 0$)	$U_{21} = 0$ ($p_x > 0$)	General expression	$U_{21} = 1$ ($p_x < 0$)	$U_{21} = U_{20}$ ($p_x = 0$)	$U_{21} = 0$ ($p_x > 0$)
0.25	$0.486–0.09\ b_1\xi$	0.495	0.486	0.473	$0.472–0.018\ b_1\xi$	0.490	0.472	0.468
1	$0.491–0.005\ b_1\xi$	0.496	0.491	0.484	$0.482–0.011\ b_1\xi$	0.493	0.482	0.479
4	$0.495–0.003\ b_1\xi$	0.498	0.495	0.490	$0.490–0.006\ b_1\xi$	0.496	0.490	0.488

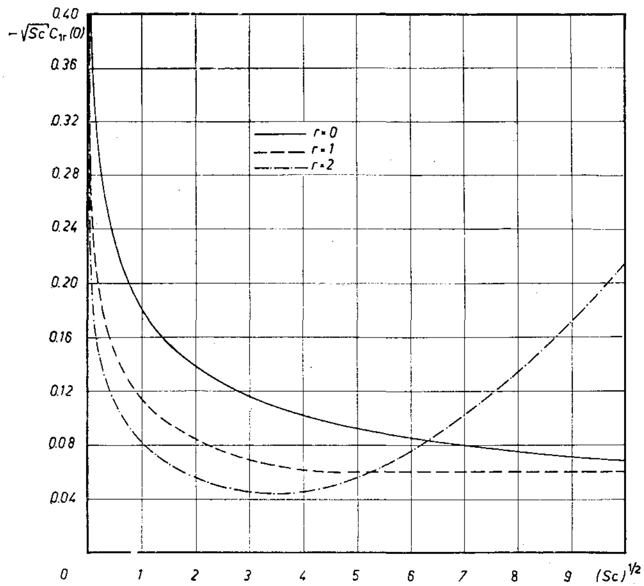

Fig. 4 Concentration at the dividing streamline; influence of Sc on first-order terms.

and the ratio between M and the mass M_0 diffused in isobaric conditions is, to within terms of order m,

$$\frac{M}{M_0} = (\xi/x)^{1/2}\left[1 - \frac{mb_1\xi}{8}\right] = [(U_1)_m]^{1/2}\left[1 - \frac{mb_1\xi}{8}\right]$$

where $(U_1)_m = \int_0^x U_1(t)dt$ is the average value of $U_1(\xi)$ over the interval $(0, x)$. Thus, the ratio (M/M_0) is influenced by the pressure gradient, both through the value of $[(U_1)_m]$ $[U_{1m} \gtrless 0$ for $p_x \lesseqgtr 0]$ and through the contribution proportional to m. The latter may be of the same order of magnitude as the correction needed to go from the purely diffusive to the isobaric case. Both effects result in an increase of the ratio (M/M_0) for favorable pressure gradients.

V. Concluding Remarks

In an attempt to investigate the possible influence of pressure gradients on the characteristics of the mixing of two nonhomogeneous streams, a series solution in terms of a parameter (m) proportional to the initial momentum flux difference has been presented in the hypothesis that the field be laminar and that the variations of density and transport coefficients be negligible.

The solution pertains to some classes of analytic pressure gradients and to arbitrary values of Sc and is presented up to terms of order m in an explicit form involving classes of functions related to the complementary error function. The accuracy of the results is satisfactory for initial velocity ratios of about 0.5 when Sc is of order one. Low values of Sc improve the accuracy, whereas high values decrease it.

By analyzing the general results and applying them to an indicative class of pressure gradients, it has been found that, within the limits of the present approximation, 1) favorable pressure gradients improve the diffusion process in the sense that they increase the exchange of mass of the diffusing component; 2) the mass diffused up to a given station is always proportional to $(Sc)^{-1/2}$ independent of the pressure field; 3) the concentration along the dividing streamline is not constant for nonisobaric fields, is increased by favorable pressure gradients, and these effects tend to get stronger as Sc and the initial velocity ratio decrease; 4) the foregoing variations are usually contained within few percents, so that the constant isobaric value for the concentration along the dividing streamline may often constitute a satisfactory approximation; and 5) adverse pressure gradients act in the opposite manner.

Thus the inference is that appropriate "modulation" of the pressure field can be used to control the exchange of mass between the two streams, and that, accordingly, further studies of nonisobaric dissipative regions should be made to obtain solutions that are less limited in scope than the present one and that pertain to more realistic situations.

References

[1] Napolitano, L. G., "On some exact solutions of laminar mixing in the presence of axial pressure gradients," Polytechnic Institute of Brooklyn, PIBAL Rept. 302 (December 1955).

[2] Napolitano, L. G., "Similar solutions in compressible laminar free mixing problems," J. Aeronaut. Sci. **23,** 390 (1956).

[3] Napolitano, L. G. and Pozzi, A., "Interazione di due correnti in presenza di gradiente assiale di pressione," l'Aerotecnica **XLI,** 173–183 (1961).

[4] Toba, K., Breslan M., and Yet, K. T., "A study of some fluid mixing problems," Rensselaer Polytechnic Institute, TR AE60003 (1961).

[5] Libby, P. A., "The homogeneous boundary layer at the axisymmetric stagnation point with large rates of injection," J. Aerospace Sci. **29,** 48–60 (1962).

[6] Fox, H. and Libby, P. A., "Helium injection into the boundary layer at an axisymmetric stagnation point," J. Aerospace Sci. **29,** 921–934 (1962).

[7] Napolitano, L. G., "Interazione laminare non isobarica," Atti Fond. Politec. Mezzogiorno d'Italia **5,** 93–112 (1963).

[8] Steiger, M. H. and Bloom, M. H., "Linearized viscous free mixing with streamwise pressure gradients," General Applied Science Labs., Inc., Westbury, N. Y. (1963).

[9] Eckert, E. R. G., *Heat and Mass Transfer* (McGraw-Hill Book Co., Inc., New York, 1959).

[10] Schlichting, H., *Boundary Layer Theory* (McGraw-Hill Book Co., Inc., New York, 1955).

[11] Chapman, S. and Cowling, T. G., *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, England, 1953).

[12] Lagestrom, P. A. and Cole, J. D., "Example illustrating expansion procedure for the Navier-Stokes equations," J. Rational Mech. Analysis **41,** 817–882 (1955).

[13] Van Dyke, M., "Higher approximation in boundary layer theory, Part I," J. Fluid Mech. **14,** 161–177 (1962).

[14] Ting, L., "On the mixing of two parallel streams," J. Math. Phys. **32,** 83–101 (1960).

[15] Erdelyi, A., *Asymptotic Expansion* (Dover Publications, Inc., New York, 1956), p. 61.

[16] Napolitano, L. G., "Interazione laminare isotachia di correnti di gas diversi," Missili **2,** 5–20 (April 1960); also "On an exact solution of laminar mixing of two different gases," J. Aerospace Sci. **27,** 144–145 (1960).

[17] Hartree, D. R., "Some properties and applications of the repeated integrals of the error function," Mem. Proc. Manchester Lit. Phil. Soc. **80,** 85–102 (1935).

[18] Kaye, J., "A table of the first eleven repeated integrals of the error function," J. Math. Phys. **2,** 119–125 (1950).

[19] Napolitano, L. G., "Addizionali classi di funzioni collegate con la funzione complementare degli errori," Atti Fond. Politec. Mezzogiorno d'Italia **6,** 70–89 (1964).

[20] Napolitano, L. G., "Su alcune nuove classi di funzioni collegate con la funzione complementare degli errori," Missili **41,** 19–35 (October 1962).

[21] Napolitano, L. G., "The Blasius equation with three-point boundary conditions," Quart. Appl. Math. **XVI,** 397–408 (January 1959).